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Turbulent transport in the transition flow region

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Abstract—The present author's model of turbulent viscosity published in *Int. J. Heat Mass Transfer* **26**, pp. 479–508, 1983 was applied to analyse the conditions in the transition region between laminar and developed turbulent pipe flow, making use of the Rothfus' experimental data from *Chem. Eng. Prog.* **4**, pp. 533–538, 1953. Theoretical value of the critical Reynolds number $Re_{crit} = 2295.18$ for transition from laminar to turbulent pipe flow was determined from the limit values of turbulent transport characteristics.

1. INTRODUCTION

IN [1], A MODEL of eddy viscosity was developed from the idea that the turbulent transport is caused by two stochastic processes. The course of the eddy viscosity is assumed to be similar to Rayleigh distribution of probability density of the reach of these processes into the flow. The following relation for the relative eddy viscosity is given in [1]

$$\frac{\varepsilon}{\nu} = \frac{2A}{\alpha}\left\{Y\left[\exp\left(-\frac{Y^2}{\alpha}\right)-\exp\left(-\frac{Y^2}{\alpha_w}\right)\right] +(2-Y)\left[\exp\left(-\frac{(2-Y)^2}{\alpha}\right) -\exp\left(-\frac{(2-Y)^2}{\alpha_w}\right)\right]-(2+Y) \times\left[\exp\left(-\frac{(2+Y)^2}{\alpha}\right)-\exp\left(-\frac{(2+Y)^2}{\alpha_w}\right)\right] -(4-Y)\left[\exp\left(-\frac{(4-Y)^2}{\alpha}\right) -\exp\left(-\frac{(4-Y)^2}{\alpha_w}\right)\right]\right\}. \tag{1}$$

In the expression for eddy viscosity the terms containing the coordinate Y represent the basic effect of the wall. For symmetry reasons the terms containing $(2-Y)$ were included. Small corrective terms containing $(2+Y)$ and $(4-Y)$ and representing negative mirror-like functions outside the range $\langle 0,2\rangle$, which are practically ineffective in the transition flow region computations and can be neglected, were introduced to obtain the exact zero values on the wall. The stochastic processes considered have the relative average reach into the flow

$$\eta = \tfrac{1}{2}\sqrt{(\pi\alpha)}, \tag{2a}$$

$$\eta_w = \tfrac{1}{2}\sqrt{(\pi\alpha_w)}. \tag{2b}$$

The dispersions σ_Y^2 and σ_{Yw}^2 of these processes are considered to be conformable to the Zhukowsky numbers Zh_σ and $Zh_{\sigma w}$, representing the time scale of the eddy elements

$$\sigma_Y^2 = Zh_\sigma = \frac{4-\pi}{4}\alpha, \tag{3a}$$

$$\sigma_{Yw}^2 = Zh_{\sigma w} = \frac{4-\pi}{4}\alpha_w. \tag{3b}$$

2. TRANSITION FLOW REGION

The region of flow transition has been analysed by a procedure identical with that of the preceding paper [1], where the coefficients in the model of eddy viscosity were determined for the fully developed pipe flow region within the range $Re \in \langle 10^4, 10^6\rangle$. Values of Fanning friction factor f, velocity U_0 in the centre of the pipe after Rothfus [2], and the condition of minimum of mean value of relative eddy viscosity along the pipe radius, $(\varepsilon/\nu)_s = A(1-\alpha_w/\alpha) = A_t$ were used in the determination of the coefficients A, α, and α_w in the expression (1). At $Re = 10^4$, there is not an exact agreement between the Rothfus' data and the data of Nikuradse, used in [1]. The initial values of f and U_0 for particular Reynolds numbers are presented in Table 1 together with the computed values of coefficients A, α, and α_w. The table also includes the coordinates of the inflexion point of eddy viscosity, Y_{in}, determined from the condition $d^2(\varepsilon/\nu)/dY^2 = 0$, dimensionless coordinate of the inflexion point, $y^+_{in} = \sqrt{(f/2)}(Re/2)Y_{in}$, the location of the point where the eddy viscosity ε equals the kinematic viscosity ν, $Y_{\varepsilon=\nu}$ and $y^+_{\varepsilon=\nu}$, the turbulent energy production rate, W_t, determined from the relation (4)

$$W_t = \frac{\frac{\varepsilon}{\nu}\left(\frac{R}{\nu+\varepsilon}\right)^2}{\nu}, \tag{4}$$

the location of the maximum production of turbulent energy, $Y_{t\max}$ and $y^+_{t\max}$, determined from the condition $dW_t/dY = 0$, and the value of the coefficient of

NOMENCLATURE

A	coefficient in the expression for eddy viscosity
R	relative radius, r/r_w, or $1-Y$
U	relative local velocity, u/u_s
W_t	turbulent energy production rate
Y	relative coordinate, y/r_w, or $1-R$
Ho	criterion of homochronicity, $u_s t/r_w$
Re	Reynolds number, $2u_s r_w/v$
Zh	Zhukowsky number, vt/r_w^2
a, b	coefficients
f	Fanning friction factor
q	heat flux density
r	radius
t	time
u	velocity in the x direction
u^*	friction velocity, $\sqrt{(\tau_w/\rho)}$
u^+	dimensionless velocity, u/u^*
x, y	coordinates
y^+	dimensionless coordinate, u^*y/v.

Greek symbols

α	coefficient in the expression for eddy viscosity
v	kinematic viscosity
τ	shear stress
ρ	density of mass
σ^2	dispersion
η	mean reach of a stochastic process
ε	eddy viscosity.

Subscripts

Y	related to the relative coordinate
0	valid for the centre line
s	mean, bulk
t	turbulent
w	related to the wall
σ	corresponding to dispersion
in	inflexion
crit	critical
max	maximum.

proportionality of eddy viscosity near the wall to the third power of the wall distance Y, $K = 2A_t/\alpha\alpha_w$

$$\frac{\varepsilon}{v} = KY^3. \tag{5}$$

In Fig. 1, the course of the relative total viscosity $(v+\varepsilon)/v$ is displayed depending on the relative wall distance Y for individual values of the Reynolds number Re. Proportionality of eddy viscosity near the wall to the third power of the wall distance can be seen

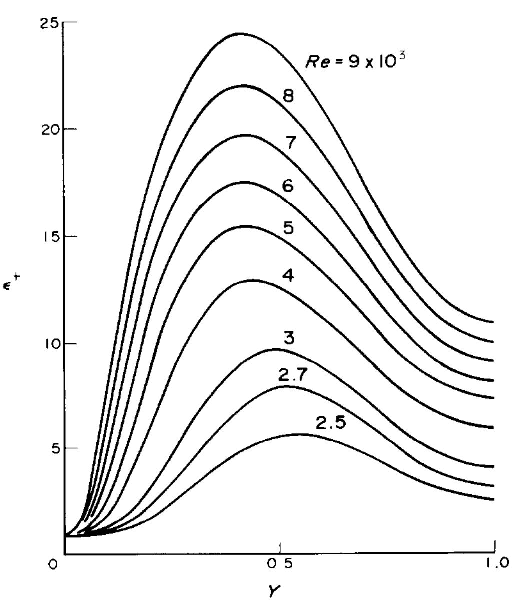

FIG. 1. Total relative viscosity.

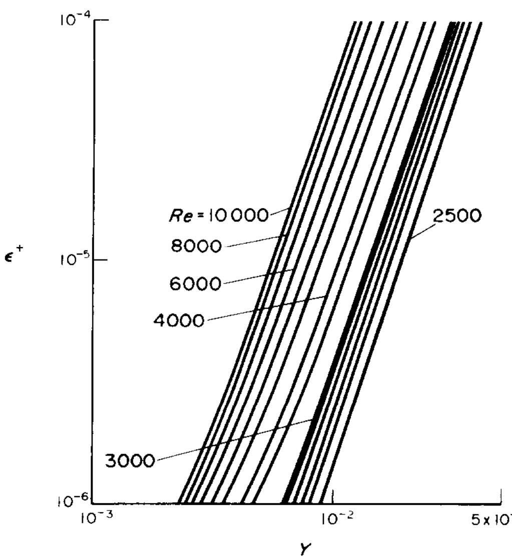

FIG. 2. Dimensionless eddy viscosity near the wall.

in Fig. 2, where the course of the dimensionless eddy viscosity

$$\varepsilon^+ = (\varepsilon/v)\frac{1}{\sqrt{f/2}(\mathrm{Re}/2)}$$

is plotted depending on the dimensionless wall distance $y^+ = \sqrt{f/2}(\mathrm{Re}/2)Y$ near the wall. Figure 3 shows computed dimensionless profiles $u^+ = f(y^+, Re)$. Local velocities were determined from the relation

$$U = \frac{f}{4}Re\int_0^R \frac{R}{\frac{v+\varepsilon}{v}}\,\mathrm{d}R \tag{6}$$

Table 1. Characteristics of transition flow region

Re	U_0	f	$\frac{f}{4}Re$	$\sqrt{\frac{f}{2}}\frac{Re}{2}$	A	α	α_w	A_t	K	Y_{in}	y^+_{in}
2500	1.4992	0.0100	6.25	8.8388×10^1	92.28702	0.198200	0.1929	2.46776	1.29091×10^2	0.31267	27.6361
2600	1.4599	0.0103	6.69	9.3292×10^1	13.03940	0.213560	0.1623	3.12978	1.80595×10^2	0.30434	28.3922
2700	1.4347	0.0104	7.02	9.7349×10^1	7.62815	0.248952	0.1294	3.66321	2.27426×10^2	0.29533	28.7502
2800	1.4184	0.0105	7.35	1.0143×10^2	6.51370	0.276155	0.1031	4.08305	2.86955×10^2	0.28121	27.3437
2900	1.4084	0.0105	7.61	1.0506×10^2	6.26567	0.291533	0.8773×10^{-1}	4.38016	3.42519×10^2	0.26958	28.1917
3000	1.3947	0.0104	7.80	1.0816×10^2	6.77107	0.293974	0.8613×10^{-1}	4.78720	3.78136×10^2	0.26834	29.0236
3500	1.3680	0.0102	8.92	1.2497×10^2	7.01502	0.326238	0.5018×10^{-1}	5.93601	7.25202×10^2	0.22544	28.1735
4000	1.3441	0.0098	9.80	1.4000×10^2	8.11767	0.334522	0.3958×10^{-1}	7.15721	1.08112×10^3	0.20670	28.9378
4500	1.3298	0.0096	10.80	1.5588×10^2	9.01077	0.340530	0.2923×10^{-1}	8.23731	1.65513×10^3	0.18370	28.6354
5000	1.3193	0.0092	11.50	1.6955×10^2	9.81785	0.342525	0.2478×10^{-1}	9.10758	2.14605×10^3	0.17181	29.1298
6000	1.3072	0.0087	13.05	1.9786×10^2	11.20872	0.344582	0.1686×10^{-1}	10.66029	3.66985×10^3	0.14607	28.9018
7000	1.2987	0.0084	14.70	2.2682×10^2	12.68224	0.344407	0.1179×10^{-1}	12.24810	6.03268×10^3	0.12482	28.3126
8000	1.2920	0.0082	16.40	2.5612×10^2	14.24528	0.343217	0.8570×10^{-2}	13.88958	9.44424×10^3	0.10803	27.6697
9000	1.2853	0.0080	18.00	2.8460×10^2	15.87842	0.341910	0.6677×10^{-2}	15.15683	1.32784×10^4	0.09626	27.3966
10000	1.2820	0.0077	19.25	3.1024×10^2	17.07024	0.340775	0.5526×10^{-2}	16.79433	1.78357×10^4	0.08810	27.3326

Re	W_{tmax}	Y_{tmax}	y^+_{tmax}	$Y_{\varepsilon=\nu}$	$y^+_{\varepsilon=\nu}$	η	η_w	$Zh_\sigma = \sigma_Y^2$	$\frac{2}{Zh_\sigma}$	Ho_σ	$Zh_{\sigma w} = \sigma_{Yw}^2$	$\frac{2}{Zh_{\sigma w}}$	$Ho_{\sigma w}$
2500	0.2278	0.18679	16.5103	0.21392	18.9078	0.3945	0.3892	0.0425341	47.0448	53.1408	0.0413967	48.3130	51.7459
2600	0.2288	0.16832	15.7032	0.18866	17.6007	0.4095	0.3570	0.0458303	43.6392	59.5795	0.0348299	57.4220	45.2788
2700	0.2298	0.15678	15.2624	0.17377	16.9166	0.4421	0.3187	0.0534257	37.4356	72.1239	0.0277695	72.0215	37.4888
2800	0.2311	0.14607	14.8164	0.16048	16.2775	0.4657	0.2845	0.0592634	33.5483	82.9672	0.0221254	90.3936	30.9756
2900	0.2320	0.13841	14.5413	0.15116	15.8806	0.4785	0.2624	0.0625635	31.9678	90.7162	0.0188270	106.2303	27.2992
3000	0.2323	0.13404	14.4998	0.14578	15.7673	0.4805	0.2601	0.0630874	31.7025	94.6297	0.0184837	108.2037	27.7255
3500	0.2355	0.10967	13.7054	0.11724	14.6510	0.5062	0.1985	0.0700114	28.5667	122.5205	0.0107687	185.7231	18.8453
4000	0.2371	0.09659	13.5221	0.10227	14.3185	0.5126	0.1763	0.0717891	27.8596	143.5772	0.0084939	235.4620	16.9879
4500	0.2385	0.08442	13.1589	0.08868	13.8238	0.5172	0.1515	0.0730784	27.3679	164.4263	0.0062782	318.8363	14.1138
5000	0.2395	0.07767	13.1695	0.08124	13.7738	0.5187	0.1395	0.0735065	27.2089	183.7636	0.0053198	376.0930	13.2946
6000	0.2412	0.06545	12.9504	0.06794	13.4438	0.5220	0.1151	0.0739480	27.0462	221.8425	0.0036182	552.7630	10.8546
7000	0.2425	0.05581	12.6586	0.05760	13.0658	0.5220	0.0962	0.0739108	27.0596	258.6886	0.0025302	790.4652	8.8555
8000	0.2436	0.04830	12.3709	0.04984	12.7128	0.5191	0.0820	0.0736554	27.1542	294.5160	0.0018391	1087.4661	7.3565
9000	0.2443	0.04286	12.1996	0.04391	12.4969	0.5182	0.0724	0.0733745	27.2574	330.1853	0.0014329	1395.7774	6.4480
10000	0.2448	0.03929	12.1903	0.04017	12.4618	0.5173	0.0659	0.0731310	27.3486	365.6493	0.0011859	1686.4974	5.9294

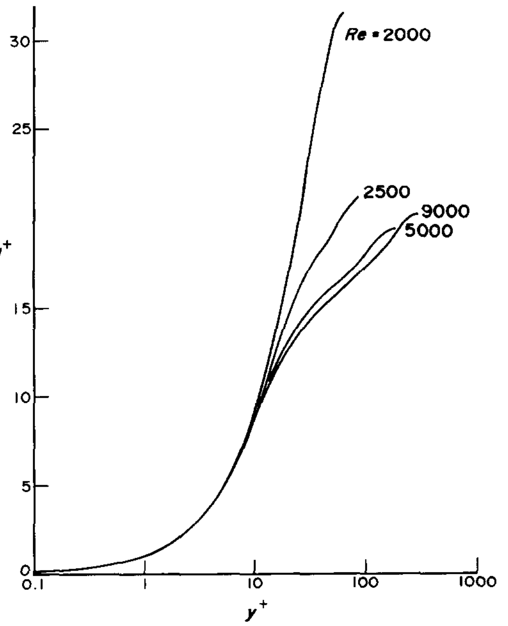

FIG. 3. Dimensionless velocity profiles.

and further transformed into a dimensionless form [1]

$$u^+ = \frac{U}{\sqrt{(f/2)}}. \tag{7}$$

The courses of the Reynolds stress expressed by the relative value

$$\tau_t/\tau_w = R\left[1 - \frac{1}{(v+\varepsilon)/v}\right]$$

are plotted in Fig. 4. For practical application, substitute relationships for the coefficients A, α, and α_w have been devised from their values at the discrete Reynolds numbers

$$A = \frac{Re}{\sum_{n=0}^{n=7} a_n Re^n}, \quad Re \in \langle 2600, 10000 \rangle \tag{8}$$

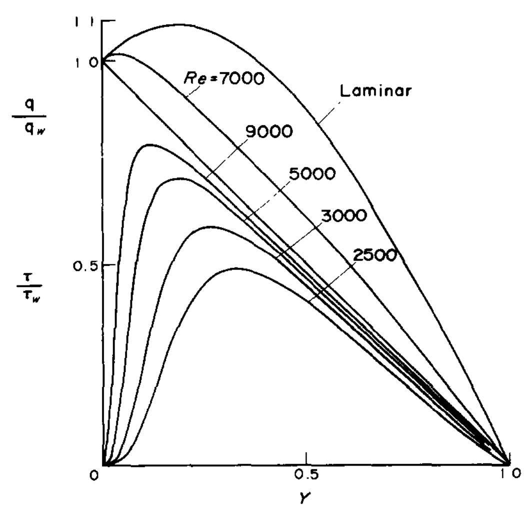

FIG. 4. Reynolds turbulent shear stress τ/τ_w and the ratio q/q_w.

$a_0 = -1.410632 \times 10^4$ $\quad a_4 = -1.447000 \times 10^{-10}$

$a_1 = 1.456988 \times 10^1$ $\quad a_5 = 9.706550 \times 10^{-15}$

$a_2 = -5.841680 \times 10^{-3}$ $\quad a_6 = -3.426050 \times 10^{-19}$

$a_3 = 1.222892 \times 10^{-6}$ $\quad a_7 = 4.917500 \times 10^{-24}$

$$\alpha = \frac{Re}{\sum_{n=0}^{n=7} b_n Re^n} \qquad Re \in \langle 2500, 10000 \rangle \tag{9}$$

$b_0 = 1.296640 \times 10^5$ $\quad b_4 = 1.139080 \times 10^{-9}$

$b_1 = -1.207270 \times 10^2$ $\quad b_5 = -7.546700 \times 10^{-14}$

$b_2 = 4.794600 \times 10^{-2}$ $\quad b_6 = 2.639900 \times 10^{-18}$

$b_3 = -9.794400 \times 10^{-6}$ $\quad b_7 = -3.764500 \times 10^{-23}$

$$\alpha_w = \frac{4440}{(Re-1650)^{3/2}} \qquad Re \in \langle 2600, 10000 \rangle. \tag{10}$$

The coefficient α reaches its maximum value at $Re \sim 6 \times 10^3$. The value of the coefficient α_w strongly increases with the decrease of the Reynolds number, till in the pure laminar flow equals the coefficient α. The coefficient A_t increases with Reynolds number. The coefficient A has minimum at $Re \sim 2.9 \times 10^3$. At small Reynolds numbers, the coefficient A increases rapidly with the decrease of Re. In laminar flow, the coefficient A would reach any arbitrary value from the point of view of mathematics considering that owing to the equality $\alpha = \alpha_w$ the effects of both particular stochastic processes do not manifest themselves. The change of nature of the turbulent transport is given by the coordinate of the inflexion point Y_{in}, which, as can be seen in Table 1, is the dimensionless form y^+_{in} practically constant ($y^+_{in} = 28 \div 29$) throughout all transition region of the flow. Considering only the basic terms containing the coordinate Y the following limit for $\alpha = \alpha_w$ follows for the eddy viscosity inflexion point coordinate from the second derivative of (1):

$$\lim_{\alpha_w \to \alpha} Y_{in} = \sqrt{\left(\frac{3}{2}\frac{\alpha}{2}\right)}. \tag{11}$$

3. CONDITION OF TRANSITION FROM LAMINAR TO TURBULENT FLOW

We shall start from a consideration, that the inception of turbulent motion or stochastic processes in laminar flow is initiated at the point, where the conditions of transverse transport of momentum change their nature. For transverse transport in a flow of a medium the ratio q/q_w, defined by the relation

$$q/q_w = \frac{2}{R}\int_0^R UR \, dR, \tag{12}$$

plays the decisive part. Its nature changes at the point, where it acquires the value $q/q_w = 1$, conformable to the value corresponding to the pipe total cross section:

$$\frac{2}{R_1}\int_0^{R_1} UR \, dR = 1. \tag{13}$$

The reason for use of the notation q/q_w lies in the fact, that for the nonisothermal flow with constant heat flux density on the wall, q_w = const., the relation (12) gives the distribution of the local heat flux density over the pipe cross section [3]. The course of the ratio q/q_w for $Re = 7000$ and laminar flow is plotted in Fig. 4.

In laminar flow, where the velocity is expressed by the relation

$$U = 2(1-R^2) \tag{14}$$

the ratio q/q_w is

$$q/q_w = 2R-R^3. \tag{15}$$

The point where $q/q_w = 1$ is given by the equation (16):

$$R^3-2R+1 = 0 \tag{16}$$

with the following roots in the range $\langle 0, 1 \rangle$:

$R_1 = 1$, and

$$R_1 \doteq 0.618033988 \; (Y_1 = 1-R_1 = 0.381966012).$$

We shall situate the inflexion point Y_{in} of eddy viscosity in the limit $\alpha = \alpha_w$ to the point Y_1 where the conditions of transverse transport in laminar flow change their nature. Substituting $Y_1 = Y_{in}$ into (11) leads to the limit value $\alpha = \alpha_w = 0.1945307$. The values α in Table 1 approach this value from above, the values α_w from below. This limit value of coefficient $\alpha = \alpha_w$ characterising the inception of random disturbances holds throughout whole laminar flow region. Knowledge of this value allows us to determine also further characteristics of initiated disturbances, their dimensionless time scale $Zh_\sigma = Zh_{\sigma w} = 0.041747$ among others. Constant characteristics of turbulent disturbances in laminar flow indicate the possibility to express the conditions of transition from laminar to turbulent flow in different form than by means of critical Reynolds number [3, 4, 5] $Re_{crit} \sim 2300$ used in the present practice.

By formal rearrangement of the expression for Reynolds number and its extension by the time scale of disturbances t_σ, the following expression is obtained:

$$Re = \frac{2r_w u_s}{\nu} = \frac{2r_w^2}{\nu t_\sigma}\frac{u_s t_\sigma}{r_w} = \frac{2}{Zh_\sigma}Ho_\sigma. \tag{17}$$

When the Reynolds number is expressed as a product of two criteria, the first criterion characterising turbulent disturbances by twice the reciprocal value of the Zhukowsky number for laminar flow $2r_w^2/(\nu t_\sigma)$ is constant and has the value of 47.908. The nature of the flow must be therefore determined by the value of the second criterion, $Ho_\sigma = u_s t_\sigma / r_w$, which is in the theory of similarity usually called criterion of homochronicity and which is a measure of expansion of disturbances in the flow. When $Ho_\sigma < 2/Zh_\sigma$ the flow is laminar. Transition into turbulent flow occurs at $Ho_\sigma = 2/Zh_\sigma$. This condition leads to the value of critical Reynolds number

$$Re_{crit} = \left(\frac{2}{Zh_\sigma}\right)^2 = 47.908^2 = 2295.18. \tag{18}$$

For turbulent flow the condition $Re > Re_{crit}$ can be replaced by the equivalent inequality $Ho_\sigma > 2/Zh_\sigma$.

Mutual relations between the criteria of homochronicity Ho_σ or $Ho_{\sigma w}$, and the criteria $2/Zh_\sigma$ or $2/Zh_{\sigma w}$ in developed turbulent flow and also in the transition region are a measure of stability of the involved random processes in the core and near the wall. The values of criteria $2/Zh_\sigma$, $2/Zh_{\sigma w}$, Ho_σ, and $Ho_{\sigma w}$ are in Table 1.

4. CONCLUSION

Values of coefficients in the expression for the eddy viscosity, which were derived from analysis of transition region between laminar and developed turbulent pipe flow, and the limit values of these coefficients allow us to determine the time scale of turbulent disturbances in laminar flow. This time scale is decisive for the flow transition. On its basis, it is possible to determine the theoretical value of critical Reynolds number $Re_{crit} = 2295.18$. Relations between similarity criteria $Ho_\sigma = u_s t_\sigma / r_w$, or $Ho_{\sigma w}$, and $2/Zh_\sigma$ or $2/Zh_{\sigma w}$ are a measure of stability of vortex processes in the core and the wall region of the flow.

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TRANSPORT TURBULENT DANS LA REGION D'ECOULEMENT DE TRANSITION

Résumé—Le modèle basé sur la viscosité turbulente et présenté par l'auteur dans *Int. J. Heat Mass Transfer*, **26**, 479–508, 1983, est appliqué à l'analyse des conditions de transition entre laminaire et turbulent dans un tube, en utilisant les données expérimentales de Rothfus dans *Chem. Engng Prog.* **4**, 533–538, 1953. La valeur théorique du nombre de Reynolds critique $Re_{crit} = 2\,295{,}18$ pour la transition est déterminée à partir des valeurs limites des caractéristiques du transport turbulent.

TURBULENZDIFFUSION IN DER UNENTWICKELTEN STRÖMUNG

Zusammenfassung—Das in *Int. J. Heat Mass Transfer* **26,** pp. 479–508, 1983 veröffentlichtes Turbulenzzähigkeitsmodell des Autors ist appliziert zur Analyse der unentwickelten Strömung in glatten Rohren auf dem Grunde der Rothfus's experimentalen Daten von *Chem. Eng. Prog.* **49,** 1953. Aus den Limitwerten der Turbulenzcharakteristiken ist der theoretische Wert der kritischen Reynoldszahl für die Grenze zwischen der laminaren und turbulenten Strömung $Re_{\mathrm{krit}} = 2295.18$ abgeleitet.

ТУРБУЛЕНТНЫЙ ПЕРЕНОС В ПЕРЕХОДНОЙ ОБЛАСТИ ТЕЧЕНИЯ

Аннотация—Модель турбулентной вязкости, предложенная автором в ранее опубликованной работе [*Int. J. Heat Mass Transfer* **26**, 479–508 (1983)], использована для анализа переходной области между ламинарным и развитым турбулентным течениями в трубе на основе экспериментальных данных Ротфуса [*Chem. Engng Prog.* **49**, 533–538 (1983)]. Из предельных величин характеристик турбулентного переноса теоретически найдена величина критического числа Рейнольдса перехода ламинарного течения в трубе в турбулентное, $Pe_{\mathrm{кр}} = 2295{,}18$.